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Abstract: In 1974 P. Lax introduced an algebro-analytic mechanism similar to the Lax L-A pair. Using it we prove global existence and uniqueness for solutions of the initial value problem for mKdV in classes of smooth functions which can be unbounded at infinity, and may even include functions which tend to infinity with respect to the space variable. Moreover, we establish the invariance of the spectrum and the unitary type of the Schrödinger operator under the KdV flow and the invariance of the spectrum and the unitary type of the impedance operator under the mKdV flow for potentials in these classes.

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Solutions of mKdV in classes of functions unbounded at infinity

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January 11, 2006

Abstract

We prove global (in time) existence and uniqueness theorems for solutions of the modified Korteweg - de Vries equation (mKdV) in classes of smooth functions which are unbounded at infinity.

1 Introduction

The purpose of this work is to solve the *modified Korteweg - de Vries equation* (mKdV) on the line

$$r_t - 6r^2 r_x + r_{xxx} = 0 \tag{1}$$

$$r|_{t=0} = r_0 \tag{2}$$

in various classes of smooth functions (possibly) unbounded at $+\infty$ and/or $-\infty$. Equation (1) is closely related to the celebrated Korteweg - de Vries equation (KdV),

$$q_t - 6qq_x + q_{xxx} = 0 \tag{3}$$

and is a model equation for wave propagation.

Let $I = (a, b) \subseteq \mathbb{R}$ with $-\infty \leq a < b \leq \infty$. For any given $\beta \in \mathbb{R}$ denote by $\mathcal{S}_\beta(I\times\mathbb{R})$ the linear space of $C^\infty(I\times\mathbb{R})$ functions having asymptotic expansions at $+\infty$ and $-\infty$ (cf. [2])

$$r(t, x) \sim \sum_{k=0}^{\infty} a_k^+(t) x^{\beta_k} \ \text{ as } \ x \to \infty \tag{4}$$

and

$$r(t, x) \sim \sum_{k=0}^{\infty} a_k^-(t) (-x)^{\beta_k} \ \text{ as } \ x \to -\infty \tag{5}$$

where $a_k^\pm \in C^\infty(I)$ and $\beta = \beta_0 > \beta_1 > \ldots$ with $\lim_{k\to\infty} \beta_k = -\infty$. By definition, the relations (4) and (5) mean that for any compact interval $J \subseteq I$ and any

$N \geq 0$, $i, j \geq 0$, there exists a constant $C_{J,N,i,j} > 0$ such that for any $\pm x \geq 1$ and $t \in J$

$$\Big|\, \partial_t^i \partial_x^j \Big(r(t,x) - \sum_{k=0}^{N} a_k^{\pm}(t)(\pm x)^{\beta_k} \Big) \Big| \leq C_{J,N,i,j} |x|^{\beta_{N+1}-j}. \tag{6}$$

For an arbitrarily chosen formal series $\sum_{k=0}^{\infty} a_k^{\pm}(t)(\pm x)^{\beta_k}$, referred to as a symbol in the theory of pseudodifferential operators, there exists a function $r \in C^{\infty}(I \times \mathbb{R})$ satisfying (4) and (5) (see for example [12, Proposition 3.5]). Analogously one defines the linear space $\mathcal{S}_\beta(\mathbb{R})$ as the space of functions $r \in C^{\infty}(\mathbb{R})$ having asymptotic expansions $r(x) \sim \sum_{k=0}^{\infty} a_k^{\pm}(\pm x)^{\beta_k}$ as $x \to \pm\infty$ where $a_k^{\pm}$ are given constants, $\beta = \beta_0 > \beta_1 > \ldots$ and $\lim\limits_{k\to\infty} \beta_k = -\infty$.

In this paper we first prove the following Theorem:

Theorem 1.1. *For any $\beta < 1/2$ and for any initial data $r_0 \in \mathcal{S}_\beta(\mathbb{R})$ there exists a global in time solution $r \in \mathcal{S}_\beta(\mathbb{R}\times\mathbb{R})$ of the initial value problem (1)-(2). The solution r is unique in the class of solutions of (1)-(2) in $\mathcal{S}_\beta(\mathbb{R}\times\mathbb{R})$. Moreover, the coefficients $a_0^{\pm}(t)$ in the asymptotic expansion of the solution $r(t,x)$ are independent of t and are equal to the coefficients $a_0^{\pm}$ in the asymptotic expansion of the initial data r_0.*

By the same method of proof we obtain similar results for the larger spaces of functions $\mathcal{O}_\beta(I \times \mathbb{R})$ and $o_\beta(I \times \mathbb{R})$ which are (possibly) unbounded at infinity.

Let $I = (a, b) \subseteq \mathbb{R}$ with $-\infty \leq a < b \leq \infty$. For any given $\beta \in \mathbb{R}$ denote by $\mathcal{O}_\beta(I \times \mathbb{R})$ the linear space of functions $r(t,x)$ in $C^{\infty}(I \times \mathbb{R})$ such that for any compact interval $J \subseteq I$ and any $k, l \geq 0$ there exists a constant $C_{J,k,l} > 0$ such that for any $|x| \geq 1$ and any $t \in J$

$$|\partial_t^k \partial_x^l r(t,x)| \leq C_{J,k,l} |x|^{\beta - l}.$$

Analogously one defines the linear space $\mathcal{O}_\beta(\mathbb{R})$ as the space of functions $r(x)$ in $C^{\infty}(\mathbb{R})$ such that for any $l \geq 0$ there exists $C_l > 0$ such that for any $|x| \geq 1$, $|\partial_x^l r(x)| \leq C_l |x|^{\beta - l}$.

We will also consider the following spaces. For any given $\beta \in \mathbb{R}$ denote by $o_\beta(I \times \mathbb{R})$ the linear space of functions $r(t,x)$ in $C^{\infty}(I \times \mathbb{R})$ such that for any compact interval $J \subseteq I$ and any $k, l \geq 0$

$$\partial_t^k \partial_x^l r(t,x) = o(|x|^{\beta - l})$$

uniformly in $t \in J$. In the same way as above one defines the space $o_\beta(\mathbb{R})$. Clearly the following inclusions hold:

$$\mathcal{S}_\beta(I \times \mathbb{R}) \subseteq \mathcal{O}_\beta(I \times \mathbb{R}), \quad o_\beta(I \times \mathbb{R}) \subseteq \mathcal{O}_\beta(I \times \mathbb{R}).$$

Theorem 1.2. *For any $\beta < 1/2$ and for any initial data $r_0 \in \mathcal{O}_\beta(\mathbb{R})$ there exists a global in time solution $r \in \mathcal{O}_\beta(\mathbb{R}\times\mathbb{R})$ of the initial value problem (1)-(2). The solution r is unique in the class of solutions of (1)-(2) in $\mathcal{O}_\beta(\mathbb{R}\times\mathbb{R})$.*

Theorem 1.3. *For any $\beta \leq 1/2$ and for any initial data $r_0 \in o_\beta(\mathbb{R})$ there exists a global in time solution $r \in o_\beta(\mathbb{R} \times \mathbb{R})$ of the initial value problem (1)-(2). The solution r is unique in the class of solutions of (1)-(2) in $o_\beta(\mathbb{R} \times \mathbb{R})$.*

Remark 1.4. *Note that for $r_0 \in \mathcal{S}_\beta(\mathbb{R})$ with $\beta = \beta_0 > 1/2$, with an asymptotic expansion of the form*

$$r_0(x) \sim \sum_{k=0}^{\infty} a_k^+ x^{\beta_k} \quad as \quad x \to +\infty$$

with $a_0^+ \neq 0$, no formal solution and therefore no solution of mKdV in $\mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$ exists.

Remark 1.5. *In fact, the uniqueness in all Theorems 1.1, 1.2, 1.3 holds if we only require $r \in o_{1/2}(\mathbb{R} \times \mathbb{R})$ which is the largest class where the existence is claimed in these theorems. So if we only require $r \in o_{1/2}(\mathbb{R} \times \mathbb{R})$ and take the initial condition r_0 in $\mathcal{S}_\beta(\mathbb{R})$, $\mathcal{O}_\beta(\mathbb{R})$ ($\beta < 1/2$) or $o_\beta(\mathbb{R})$ ($\beta \leq 1/2$), then we will automatically have $r \in \mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$, $\mathcal{O}_\beta(\mathbb{R} \times \mathbb{R})$ or $o_\beta(\mathbb{R} \times \mathbb{R})$ respectively.*

Similar results for KdV as the ones stated for mKdV in Theorem 1.1, 1.2, and 1.3 have been obtained in a series of papers by Bondareva - Shubin and Bondareva [1, 2, 3, 4] – see Appendix B where for the convenience of the reader, we give a short summary of these results. In fact, we construct our solutions of mKdV with the properties stated in the above theorems by applying to the solutions of Bondareva - Shubin an inverse of the Miura map. Recall that the Miura map $r \mapsto B(r) := r_x + r^2$, first introduced in [11], maps smooth solutions of mKdV to smooth solutions of KdV. However, the Miura map is usually neither 1-1 nor onto. This is, for example, the case when B is considered as a map $H^\beta_{loc}(\mathbb{R}) \to H^{\beta-1}_{loc}(\mathbb{R})$ with $\beta \geq 0$ [7]. In this case, the preimage of an element in $H^{\beta-1}_{loc}(\mathbb{R})$ is either the empty set, a point or a set homeomorphic to an interval. To describe the preimage $B^{-1}\{B(r)\}$ of $q = B(r)$, note that the positive function $\psi(x) = e^{\int_0^x r(s)ds}$ satisfies

$$-\psi_{xx} + (r_x + r^2)\psi = 0 \tag{7}$$

and is related to r by $r = \psi_x/\psi$. It has been shown in [7] that for $r \in H^\beta_{loc}(\mathbb{R})$ given with $\beta \geq 0$, any function in the preimage $B^{-1}\{B(r)\}$ arises in this way, i.e. for any $r \in H^\beta_{loc}(\mathbb{R})$,

$$B^{-1}\{B(r)\} = \{\psi_x/\psi \mid \psi \in H^\beta_{loc}(\mathbb{R}) \text{ positive, satisfying (7)}\}.$$

Given initial data r_0 in the class of functions considered in the theorems above, $q_0 = B(r_0)$ has the growth condition at infinity required by the theorems in [1, 2, 3, 4] to conclude that there exists a unique solution $q(t, x)$ of KdV in the corresponding class with $q(0, \cdot) = q_0$. We then consider the linear evolution equation

$$\psi_t(t, x) = Q(t)\psi(t, x)$$
$$\psi(0, x) = e^{\int_0^x r_0(s)ds}$$

where $Q(t)$ is the first order differential operator,

$$Q(t) := 2q(t,x)\partial_x - q_x(t,x). \tag{8}$$

and prove that there exists a unique, globally (in time) defined solution $\psi(t,x)$, satisfying $\psi(t,x) > 0$ for any $x \in \mathbb{R}$, $t \in \mathbb{R}$ and

$$-\psi_{xx}(t,x) + q(t,x)\psi(t,x) = 0. \tag{9}$$

The latter identity is shown by using the commutator relation

$$\dot{L} = [Q, L] + 4q_x L$$

where

$$L(t) := -\partial_x^2 + q(t,x). \tag{10}$$

and $\dot{L} = q_t$. The function

$$r(t,x) := \psi_x(t,x)/\psi(t,x) \tag{11}$$

is then the unique solution of mKdV with $r(0,\cdot) = r_0$ in a class of functions in $C^\infty(\mathbb{R} \times \mathbb{R})$ satisfying appropriate growth conditions. It has the claimed properties in each of the settings of Theorem 1.1, 1.2, and 1.3. We call the pair of operators (Q, L), satisfying the conditions above a Q-L pair. Such a pair allows us to construct an inverse of the Miura map and, in this way, deduce existence and uniqueness of solutions for (1) – (2) from the corresponding results for KdV.

Related work: Beside the work of Bondareva and Shubin already cited, we would like to mention earlier work on unbounded solutions of KdV by Menikoff [10] as well as work of Kenig, Ponce, and Vega [9]. Menikoff showed that for initial data in $o_1(\mathbb{R})$, KdV can be solved in $C^\infty(\mathbb{R} \times \mathbb{R})$ whereas Kenig, Ponce, and Vega studied solutions of KdV in special classes of unbounded functions, different from the ones considered in this paper. We remark that the Miura map has been used previously to obtain solutions of mKdV from solutions of KdV. In particular, we mention the paper [8] where periodic solutions of low regularity are obtained, and work of Gesztesy–Simon [5] and Gesztesy–Schweiger–Simon [6] for bounded solutions of mKdV. In [6], the existence of solutions $r(t,x)$ of mKdV is proved under the assumption that $q(t,x), q_x(t,x)$ and $q_t(t,x)$ belong to $L^\infty(\mathbb{R} \times \mathbb{R})$. Rather then solving the evolution equation (8), induced by the first order operator $Q(t)$ in (8) to obtain a representation of a solution $r(t,x)$ of the form (11), the authors of [6] use the operator $P := -4\partial_x^3 + 6q\partial_x + 3q_x$, appearing in the Lax pair for KdV. It turns out that the third order operator P is not well suited to prove the existence of a solution $r(t,x)$ of mKdV in the classes of increasing functions considered above.

2 Q-L pair

Suppose that $q \in C^\infty(\mathbb{R} \times \mathbb{R})$ and consider the differential operators $Q(t), L(t)$ given by (8) and (10), respectively.

Lemma 2.1. *The operators Q and L satisfy the following commutator relation*

$$\dot{L} = [Q, L] + 4q_x L + KdV(q) \tag{12}$$

where $\dot{L} = q_t(t, x)$ and $KdV(q) = q_t - 6qq_x + q_{xxx}$. In particular,

$$KdV(q) = 0 \quad \textit{iff} \quad \dot{L} = [Q, L] + 4q_x L. \tag{13}$$

The proof of the lemma is straightforward.

Assume that $q \in C^\infty(\mathbb{R} \times \mathbb{R})$ satisfies the KdV equation and that for any $T > 0$ there exists a constant $C_T > 0$ such that for any $|x| \geq 1$ and $t \in [-T, T]$

$$|q(t, x)| \leq C_T |x| \ . \tag{14}$$

Let $\psi_0 \in C^\infty(\mathbb{R})$ be an eigenfunction of $L(0)$ with eigenvalue 0, i.e.

$$L(0)\psi_0 = 0 \ . \tag{15}$$

Consider the equation

$$\psi_t(t, x) = Q(t)\psi(t, x) \tag{16}$$

$$\psi|_{t=0} = \psi_0. \tag{17}$$

By Lemma A.1, the initial value problem (16)-(17) has a unique solution $\psi(t, x)$ in $C^\infty(\mathbb{R} \times \mathbb{R})$.

Proposition 2.2. *If $q \in C^\infty(\mathbb{R} \times \mathbb{R})$ is a solution of KdV satisfying the growth condition (14), and $\psi(t, x) \in C^\infty(\mathbb{R} \times \mathbb{R})$ solves (16)-(17), then*

$$L(t)\psi(t, x) = 0 \ \ \forall t, x \in \mathbb{R}. \tag{18}$$

If, in addition, $\psi_0(x) > 0 \ \forall x \in \mathbb{R}$, then $\psi(t, x) > 0 \ \forall x, t \in \mathbb{R}$.

Proof. Let $\varphi(t, x) := L(t)\psi(t, x)$. It follows from (15) that $\varphi|_{t=0} = 0$. Using Lemma 2.1 and (16) one obtains

$$\begin{aligned} \varphi_t &= \dot{L}\psi + L\psi_t \\ &= ([Q, L] + 4q_x L)\psi + LQ\psi \\ &= Q(L\psi) + 4q_x(L\psi) \\ &= 2q\varphi_x + 3q_x\varphi \ . \end{aligned} \tag{19}$$

Hence $\varphi = \varphi(t, x)$ is a solution of the initial value problem

$$\begin{aligned} \varphi_t(t, x) &= 2q(t, x)\varphi_x(t, x) + 3q_x(t, x)\varphi(t, x) \\ \varphi|_{t=0} &= 0 \ . \end{aligned}$$

Applying again Lemma A.1 we obtain that $\varphi \equiv 0$.

The last statement of the proposition follows immediately from claim (b) of Lemma A.1. $\square$

Proposition 2.3. *Assume that $r \in C^\infty(\mathbb{R}\times\mathbb{R})$ is a solution of the initial value problem (1)-(2) for the mKdV equation and define*

$$\rho(t,x) := \rho_0(t) e^{\int_0^x r(t,s)\,ds} \tag{20}$$

with normalizing factor $\rho_0(t)$ given by

$$\rho_0(t) := e^{\int_0^t (2r^3 - r_{xx})|_{(\tau,0)}\,d\tau}. \tag{21}$$

Then $\psi(t,x) := \rho(t,x)$ is a solution of (16)-(17), where $Q(t) = 2q\partial_x - q_x$, $q = r_x + r^2$, and $\psi_0(x) := e^{\int_0^x r_0(s)\,ds}$. If, in addition, $q = r_x + r^2$ satisfies the growth condition (14) then $\rho(t,x)$ is the unique solution of (16)-(17) in $C^\infty(\mathbb{R}\times\mathbb{R})$.

Proof of Proposition 2.3. Using that $q = r_x + r^2$, one easily sees that $\rho(t,x)$ satisfies the equation $L(t)\rho = 0$. Differentiating the latter identity with respect to t and using Lemma 2.1 together with the fact that $q = r_x + r^2$ satisfies KdV (cf. [11]), we obtain

$$\begin{aligned} 0 &= \dot{L}\rho + L\rho_t \\ &= ([Q,L] + 4q_x L)\rho + L\rho_t. \end{aligned}$$

Using that $L(t)\rho = 0$, one then gets

$$0 = -L(Q\rho) + L\rho_t = L(\rho_t - Q\rho). \tag{22}$$

Hence, with $f(t,x) := \rho_t - Q\rho$ one has for any $t, x \in \mathbb{R}$

$$-f_{xx}(t,x) + q(t,x) f(t,x) = 0. \tag{23}$$

A direct computation shows that

$$f(t,0) = 0 \text{ and } f_x(t,0) = 0 \ \forall t \in \mathbb{R}. \tag{24}$$

By the uniqueness of the solutions of (23)-(24) for any fixed $t \in \mathbb{R}$, we conclude that $f(t,x) \equiv 0$, and therefore $\rho_t = Q\rho$. The uniqueness of the solution ρ follows from Lemma A.1 together with the assumption that $q(t,x)$ satisfies the growth condition (14). $\square$

Corollary 2.4. *Assume that $q \in C^\infty(\mathbb{R}\times\mathbb{R})$ solves the KdV equation and satisfies the growth condition (14). Let $\phi, \psi \in C^\infty(\mathbb{R}\times\mathbb{R})$ be two solutions of (16) with initial data $\phi|_{t=0} = \phi_0$ and $\psi|_{t=0} = \psi_0$ respectively where $L(0)\phi_0 = 0$ and $L(0)\psi_0 = 0$. Then the Wronskian $W(\phi,\psi) := \phi\psi_x - \psi\phi_x$ is independent of $t, x \in \mathbb{R}$.*

Proof. As $\phi(t,x)$ and $\psi(t,x)$ satisfy (18) (see Proposition 2.2) we get that the Wronskian W is independent of $x \in \mathbb{R}$. Using that $\phi_{xx} = q\phi$ and $\psi_{xx} = q\psi$ one obtains

$$\begin{aligned} W_t &= \phi_t\psi_x + \phi(\psi_t)_x - \psi_t\phi_x - \psi(\phi_t)_x \\ &= (2q\phi_x - q_x\phi)\psi_x + \phi(2q\psi_x - q_x\psi)_x - (2q\psi_x - q_x\psi)\phi_x \\ &\quad -\psi(2q\phi_x - q_x\phi)_x \\ &= 0\,. \end{aligned}$$

□

Theorem 2.5. *Consider the initial value problem (1)-(2) for the mKdV equation with smooth initial data $r_0 \in C^\infty(\mathbb{R})$. Suppose that the solution $q = q(t,x)$ of the KdV equation (3) with the initial data $q|_{t=0} = q_0 := r_0' + r_0^2$ is defined globally in time, $q \in C^\infty(\mathbb{R}\times\mathbb{R})$, and satisfies the growth condition (14). Then*

(a) the evolution equation (16)-(17) has a unique, globally defined positive solution $\psi(t,x) > 0$ and the function $r(t,x) = \psi_x(t,x)/\psi(t,x)$ is a global solution of the mKdV initial value problem (1)-(2);

(b) if $r_1, r_2 \in C^\infty(\mathbb{R}\times\mathbb{R})$ are solutions of the initial value problem of mKdV (1)-(2) both having q as their image with respect to the Miura map $r \mapsto r_x + r^2$ (i.e., $\forall t,x \in \mathbb{R}$, $r_{1x}(t,x) + r_1^2(t,x) = r_{2x}(t,x) + r_2^2(t,x)$), then $r_1 \equiv r_2$.

Remark 2.6. *Loosely speaking, statement* (*b*) *of Theorem 2.5 says that whenever KdV has a unique solution within a certain class then mKdV has a unique solution within the corresponding class defined by the Miura map.*

Proof of Theorem 2.5. (*a*) Introduce

$$\psi_0(x) = e^{\int_0^x r_0(s)\,ds}. \tag{25}$$

Clearly, $\psi_0(x) > 0\ \forall x \in \mathbb{R}$. As $q_0 = r_0' + r_0^2$ one obtains from (25) that $L(0)\psi_0 = 0$. By Proposition 2.2 the solution $\psi(t,x)$ of (16)-(17) in $C^\infty(\mathbb{R}\times\mathbb{R})$ satisfies $L(t)\psi(t,x) = 0\ \forall t,x \in \mathbb{R}$. Moreover, $\psi(t,x) > 0\ \forall t,x \in \mathbb{R}$. Consider the smooth function $r(t,x)$ given by (11). It follows from (25) that $r|_{t=0} = r_0$. Taking into account that $L(t)\psi(t,x) = 0$ one proves by a straightforward calculation that

$$\begin{aligned} mKdV(r) &:= r_t - 6r^2r_x + r_{xxx} \\ &= -(\psi_t\psi_x - \psi\psi_{xt} - 6q\psi_x^2 + 3\psi_{xx}^2 + 4\psi_x\psi_{xxx} - \psi\psi_{xxxx})/\psi^2 \end{aligned}$$

(See also formula (7.42) in [6].) Using that $\psi_t = Q\psi$ one gets that $mKdV(r) = 0$. This proves claim (*a*).

Claim (*b*) follows from Proposition 2.3, as the two solutions r_1, r_2 lead to the same operator Q (cf. (8)) and the same initial data ψ_0 (cf. (17)). Indeed, as r_1 and r_2 are solutions of (1)-(2) and $q = r_{1x} + r_1^2 = r_{2x} + r_2^2$ we get from Proposition 2.3 that for $k = 1, 2$,

$$\rho_k(t,x) := \rho_{k,0}(t) e^{\int_0^x r_k(t,s)\, ds} \quad \text{with} \quad \rho_{k,0}(t) := e^{\int_0^t (2r_k^3 - (r_k)_{xx})|_{(\tau,0)}\, d\tau}$$

are solutions of the linear initial value problem (16)-(17) with the same initial data $\psi_0(x) = \rho_k(0,x) = e^{\int_0^x r_0(s)\, ds}$. As q satisfied the growth condition (14) the solution of (16)-(17) is unique and therefore $\rho_1 \equiv \rho_2$. In particular,

$$r_1 = \frac{\rho_{1x}}{\rho_1} = \frac{\rho_{2x}}{\rho_2} = r_2 \, .$$

□

3 Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. In the sequel we will need the classes

$$\mathcal{S}^*_{\beta+1}(\mathbb{R} \times \mathbb{R}) := \{ f \in C^\infty(\mathbb{R} \times \mathbb{R}) \mid f_x \in \mathcal{S}_\beta(\mathbb{R} \times \mathbb{R}) \} \, .$$

where β is a given real number. Note that the operator of integration, $f(t,x) \mapsto \int_0^x f(t,s) ds$, maps $\mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$ to $\mathcal{S}^*_{\beta+1}(\mathbb{R} \times \mathbb{R})$ whereas the operator of differentiation, $f(t,x) \mapsto \partial_x f(t,x)$, maps $\mathcal{S}^*_{\beta+1}(\mathbb{R} \times \mathbb{R})$ to $\mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$ for any $\beta \in \mathbb{R}$. Analogously one defines $\mathcal{S}^*_{\beta+1}(\mathbb{R})$.

The following Lemma describes the functions from $\mathcal{S}^*_{\beta+1}(\mathbb{R} \times \mathbb{R})$ in terms of their asymptotics at $\pm\infty$.

Lemma 3.1. *$f \in \mathcal{S}^*_{\beta+1}(\mathbb{R} \times \mathbb{R})$ if and only if $f \in C^\infty(\mathbb{R} \times \mathbb{R})$ and it has an asymptotic expansion for $x \to \pm\infty$ of the form*

$$f(t,x) \sim \begin{cases} \sum_{k=0}^\infty a_k^\pm(t)\, (\pm x)^{\beta_k+1} + a_*^\pm(t) \log(\pm x) & \text{if} \quad \beta + 1 \geq 0 \\ \\ c^\pm(t) + \sum_{k=0}^\infty a_k^\pm(t)\, (\pm x)^{\beta_k+1} & \text{if} \quad \beta + 1 < 0 \end{cases} \tag{26}$$

where $\beta = \beta_0 > \beta_1 > \ldots$ with $\lim_{k\to\infty} \beta_k = -\infty$ and $a_k^\pm$, $a_*^\pm$, and $c_\pm$ are functions of t in $C^\infty(\mathbb{R})$. The same result holds in $\mathcal{S}^*_{\beta+1}(\mathbb{R})$.

In particular, if $\beta + 1 \geq 0$ then the leading term of the asymptotic of f is $a_0^\pm(t)(\pm x)^{\beta+1}$ (for $\beta > -1$) or $a_*^\pm(t)\log(\pm x)$ (for $\beta = -1$). If $\beta + 1 < 0$, the leading term is $c^\pm(t)$ followed by $a_0^\pm(t)(\pm x)^{\beta+1}$. The asymptotic relations should be understood similarly to (4), (5). For example, the first relation in (26) means that for any compact interval $J \subseteq \mathbb{R}$, $i, j \geq 0$, and any $N \geq 0$ with $\beta_N + 1 < 0$, there exists a constant $C_{J,N,i,j} > 0$ such that for any $|x| \geq 1$ and any $t \in J$

$$\Big| \partial_t^i \partial_x^j \Big(f(t,x) - \Big(a_*^\pm(t) \log(\pm x) + \sum_{k=0}^N a_k^\pm(t)(\pm x)^{\beta_k+1} \Big) \Big) \Big| \leq C_{J,N,i,j} |x|^{(\beta_{N+1}+1)-j}. \tag{27}$$

Proof of Lemma 3.1. If $f \in C^\infty(\mathbb{R} \times \mathbb{R})$ has an asymptotic expansion as in (26), then clearly, $f_x \in \mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$, hence $f \in \mathcal{S}^*_{\beta+1}(\mathbb{R} \times \mathbb{R})$. Let us prove the converse statement. As the asymptotic expansions for $x \to +\infty$ and $x \to -\infty$ of an element $f \in \mathcal{S}^*_{\beta+1}(\mathbb{R} \times \mathbb{R})$ are obtained in a similar way let us consider the case $x \to +\infty$ only. First we treat the case where $\beta + 1 \geq 0$. By definition, for an element $f \in \mathcal{S}^*_{\beta+1}(\mathbb{R} \times \mathbb{R})$, $f_x \in \mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$ and hence has an asymptotic expansion

$$f_x \sim \sum_{k=0}^{\infty} b_k^+(t)\, x^{\beta_k} \quad \text{as} \quad x \to \infty \tag{28}$$

where $\beta = \beta_0 > \beta_1 > \ldots$ with $\lim_{k\to\infty} \beta_k = \infty$. Without loss of generality we assume that $\beta_m = -1$ for some $m \geq 0$.[1] Formally, the claimed result is obtained by integrating term by term the right hand side of (28) with respect to the x-variable. In order to make this argument rigorous we argue as follows: For any $N \geq m+1$ and $x \in \mathbb{R}$ consider the quantity

$$Q_N(t,x) := -\chi_+(x) \int_x^\infty \Big(f_x(t,s) - \sum_{k=0}^{N} b_k^+(t) s^{\beta_k} \Big)\, ds \tag{29}$$

where $\chi_+(x)$ is a smooth cut-off function with $\chi_+(x) = 0$ for $x \leq 1/2$ and $\chi_+(x) = 1$ for $x \geq 1$. As $f_x \in \mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$ and $\beta_{m+1} < -1$ it follows that the improper integral in (29) exists and if $x \geq 1$, $\partial_x Q_N(t,x) = f_x(t,x) - \sum_{k=0}^N b_k^+(t) x^{\beta_k}$. Hence, $\partial_x Q_N$ is in $\mathcal{S}_{\beta_{N+1}}(\mathbb{R} \times \mathbb{R})$. We claim that Q_N is in $\mathcal{S}_{\beta_{N+1}+1}(\mathbb{R} \times \mathbb{R})$. To show this it remains to estimate $\partial_t^i Q_N(t,x)$. It follows from (28) that for any compact interval $J \subseteq \mathbb{R}$, $i \geq 0$, and $N \geq m+1$, there exists a constant $C_{J,N,i} > 0$ such that for any $x \geq 1$, $t \in J$

$$\begin{aligned}
|\partial_t^i Q_N(t,x)| &\leq \int_x^\infty |\partial_t^i (f_x(t,s) - \sum_{k=0}^{N} b_k^+(t) s^{\beta_k})|\, ds \\
&\leq C_{J,N,i} \int_x^\infty s^{\beta_{N+1}}\, ds \\
&\leq C_{J,N,i}\, \frac{x^{\beta_{N+1}+1}}{|\beta_{N+1}+1|}\,.
\end{aligned} \tag{30}$$

Computing the integral in (29) one gets for $x \geq 1$

$$Q_N(t,x) = f(t,x) - \Big(c^+(t) + b_m^+(t) \log x + \sum_{0 \leq k \leq N, k \neq m} \frac{b_k^+(t)}{\beta_k + 1}\, x^{\beta_k+1} \Big) \tag{31}$$

where

$$c^+(t) := f(t,1) - \sum_{k=0}^{m-1} \frac{b_k^+(t)}{\beta_k+1} + \int_1^\infty \Big(f_x(t,s) - \sum_{k=0}^{m} b_k^+(t) s^{\beta_k} \Big)\, ds\,. \tag{32}$$

[1]Take $b_m^+(t) \equiv 0$ if necessary.

(Note that the integral in (32) converges as the integrand is estimated locally uniformly in t by $O(s^{\beta_{m+1}})$ with $\beta_{m+1} < -1$.) The desired estimate (27) of $f(t,x)$ for $x \to +\infty$ follows from (28), (30), and (31).

The case $\beta + 1 < 0$ is treated in a similar way, actually it is easier than the case $\beta + 1 \geq 0$. $\Box$

Proof of Theorem 1.1. We will show that the claimed results follow from Theorem 2.5 and Lemma 3.2 stated below, combined with results of Bondareva and Shubin in [2, 1] - see Appendix B for a summary of these results. Indeed, for a given $r_0 \in \mathcal{S}_\beta(\mathbb{R})$, the Miura image $q_0 := r_{0x} + r_0^2$ belongs to $\mathcal{S}_\delta(\mathbb{R})$ with $\delta := \max\{2\beta, \beta - 1\} < 1$. According to the results in [2, 1] (cf. Theorem B.1, B.2 in Appendix B) there exists a unique solution $q \in \mathcal{S}_\delta(\mathbb{R} \times \mathbb{R})$ of the KdV equation (3) with initial data $q|_{t=0} = q_0$. As $\delta < 1$ the solution $q = q(t,x)$ satisfies the growth condition (14). In particular, according to Proposition 2.2 the linear initial value problem

$$\psi_t(t,x) = 2q(t,x)\psi_x(t,x) - q_x(t,x)\psi(t,x) \tag{33}$$

$$\psi|_{t=0} = \psi_0(x) := e^{\int_0^x r_0(s)\, ds} \tag{34}$$

has a unique (within the class of C^∞-functions) globally (in time) defined positive solution, $\psi = \psi(t,x) > 0$, which satisfies $-\psi_{xx} + q\psi = 0\ \forall t, x \in \mathbb{R}$. It follows from item (a) of Theorem 2.5 that the function $r(t,x) = \psi_x(t,x)/\psi(t,x)$ is a solution of (1)-(2). It is easy to see that the function

$$p = p(t,x) := \log \psi(t,x)$$

satisfies

$$p_t(t,x) = 2q(t,x)p_x(t,x) - q_x(t,x) \tag{35}$$

$$p|_{t=0} = p_0(x) \tag{36}$$

with initial data $p_0(x) = \int_0^x r_0(s)\, ds \in \mathcal{S}^*_{\beta+1}(\mathbb{R})$. According to Lemma 3.2 below the function $p(t,x)$ belongs to $\mathcal{S}^*_{\beta+1}(\mathbb{R} \times \mathbb{R})$ and therefore $r = \partial_x p \in \mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$.

The uniqueness of the solution $r = r(t,x)$ in the class $\mathcal{S}_\beta(\mathbb{R}\times\mathbb{R})$ follows from the uniqueness of the solution $q = q(t,x)$ in the class $\mathcal{S}_\delta(\mathbb{R} \times \mathbb{R})$ (cf. Theorem B.2 in Appendix B) and Theorem 2.5 (b). $\Box$

The proof of Theorem 1.1 used the following lemma.

Lemma 3.2. *Let $\beta < 1/2$ and $\delta := \max\{2\beta, \beta - 1\} < 1$. Consider the initial value problem (35)-(36) where $q \in \mathcal{S}_\delta(\mathbb{R} \times \mathbb{R})$. Then for any initial data $p_0 \in \mathcal{S}^*_{\beta+1}(\mathbb{R})$ there exists a solution $p \in \mathcal{S}^*_{\beta+1}(\mathbb{R} \times \mathbb{R})$ of (35)-(36). This solution is unique in $C^\infty(\mathbb{R} \times \mathbb{R})$.*

In order to prove Lemma 3.2 we will first construct formal series $\chi_\pm(t,x)$ having the form (26) and satisfying the evolution equation (35)-(36) formally for $x \to$

$\pm\infty$. As the cases $x \to +\infty$ and $x \to -\infty$ are treated in the same way we restrict our attention only to the case $x \to +\infty$.

Let $p_0 \in \mathcal{S}^*_{\beta+1}(\mathbb{R})$ be the initial data in (35)-(36). By Lemma 3.1, $p_0(x)$ has an asymptotic expansion for $x \to +\infty$ of the form

$$p_0(x) \sim \begin{cases} \sum_{k=0}^\infty p_k^+ x^{\beta_k+1} + p_*^+ \log x & \text{if} \quad \beta+1 \geq 0 \\ c^+ + \sum_{k=0}^\infty p_k^+ x^{\beta_k+1} & \text{if} \quad \beta+1 < 0 \end{cases}$$

where $\beta_0 := \beta < 1/2$ and $\beta_0 > \beta_1 > ...$, $\lim_{k\to\infty} \beta_k = -\infty$. As a solution p of (35)-(36) gives rise to the 1-parameter family of solutions $p + \texttt{const}$, we can assume without loss of generality that the constant c^+ in the asymptotic expansion for $p_0(x)$ vanishes,

$$p_0(x) \sim \sum_{k=0}^\infty p_k^+ x^{\beta_k+1} + p_*^+ \log x \quad \text{as} \quad x \to \infty\,. \tag{37}$$

Here $\beta < 1/2$ but not necessarily $\beta + 1 \geq 0$. By assumption, $q \in \mathcal{S}_\delta(\mathbb{R}\times\mathbb{R})$ and hence it has an asymptotic expansion for $x \to +\infty$ of the form

$$q(t,x) \sim \sum_{k=0}^\infty c_k^+(t) x^{\delta_k} \tag{38}$$

where $\delta_0 = \delta := \max\{2\beta, \beta-1\} < 1$ and $\delta_0 > \delta_1 > ...$, $\lim_{k\to\infty} \delta_k = -\infty$. Consider the set

$$\Delta := \{\delta_k\}_{k\geq 0}.$$

In order to find a *formal* solution $\chi_+(t,x)$ of (35)-(36) we will have to extend the set of exponents $\{\beta_k+1\}_{k\geq 0}$ appearing in (37) to a larger discrete set $\bar{B}$ with the same upper limit as $\{\beta_k+1\}_{k\geq 0}$ so that the exponents appearing in the asymptotic expansions of the left and right hand side of (35) belong to $\bar{B}$. To construct $\bar{B}$ we first need to extend the set Δ.

Lemma 3.3. *There exists an unbounded discrete set* $\bar{\Delta} \subseteq \mathbb{R}$ *with* $\Delta \subseteq \bar{\Delta}$ *such that*

(a) $\max \bar{\Delta} = \max \Delta = \delta < 1$*;*

(b) if $\delta', \delta'' \in \bar{\Delta}$ *then* $\delta' + \delta'' - 1 \in \bar{\Delta}$*;*

(c) if $\delta' \in \bar{\Delta}$ *then* $\delta' - 1 \in \bar{\Delta}$.

Proof. First note that a set $\bar{\Delta}$ satisfies (b) iff $\bar{\Delta} - 1 := \{\delta' - 1 \mid \delta' \in \bar{\Delta}\}$ satisfies

$$\delta', \delta'' \in \bar{\Delta} - 1 \quad \text{implies} \quad \delta' + \delta'' \in \bar{\Delta} - 1\,. \tag{39}$$

It is easy to see that the set $\Delta_1 \subseteq \mathbb{R}$,

$$\Delta_1 := \Big\{ \sum_{i\in J} \delta_i \mid \delta_i \in \Delta - 1, J \subseteq \mathbb{Z}_{\geq 0} \text{ is finite and } J \neq \emptyset \Big\}$$

is discrete, satisfies (39) and that $\max \Delta_1 = \delta - 1$. Consider the set

$$\bar{\Delta}_1 := \{\delta' - k \mid \delta' \in \Delta_1, k \in \mathbb{Z}_{\geq 0}\}.$$

Then

$$\delta', \delta'' \in \bar{\Delta}_1 \quad \text{implies} \quad \delta' + \delta'' \in \bar{\Delta}_1;$$

in addition, $\bar{\Delta}_1$ is unbounded and discrete. Moreover $\max \bar{\Delta}_1 = \delta - 1$, and $\delta' - 1 \in \bar{\Delta}_1$ for any $\delta' \in \bar{\Delta}_1$. Hence, the set $\bar{\Delta} := \bar{\Delta}_1 + 1$ satisfies claims (a)-(c) of the lemma. □

We extend in the sum in (38) the set of exponents Δ to $\bar{\Delta}$ by setting the new coefficients in (38) all equal to zero. Hence without loss of generality, one can – and in the sequel we will – assume that the set of the exponents $\Delta = \{\delta_k\}_{k\geq 0}$ in (38) satisfies conditions (a)-(c) of Lemma 3.3.

Let us now introduce the following subsets of $\mathbb{R}$,

$$B := \{\beta_k\}_{k\geq 0}$$

and

$$\bar{B} := \{\beta' + \delta' \mid \beta' \in B,\ \delta' \in \Delta\} \cup \Delta \cup \{\beta' + 1 \mid \beta' \in B\} \tag{40}$$

Lemma 3.4. *The set $\bar{B}$ is discrete and has the following properties:*

(i) $\max \bar{B} = \beta + 1$*;*

(ii) if $\delta' \in \Delta$ and $\beta' \in \bar{B}$, then $\delta' + \beta' - 1 \in \bar{B}$;

(iii) the set $\{\delta' - 1 \mid \delta' \in \Delta\}$ is contained in $\bar{B}$.

Proof of Lemma 3.4. The proof that $\bar{B}$ is discrete follows from the arguments used in the proof of Lemma 3.3.
(i) As $\beta < 1/2$ and $\delta = \max\{2\beta, \beta - 1\} < 1$ one gets $\max \bar{B} = \max\{\delta, \beta + \delta, \beta + 1\} = \beta + 1$.
(ii) follows from the fact that Δ has property Lemma 3.3 (b). Indeed, as any $\beta' \in \bar{B}$ can be written in the form $\beta' = \beta'' + \delta''$ ($\beta'' \in B$, $\delta'' \in \Delta$), $\beta' = \delta''$, or $\beta' = \beta'' + 1$ and as by Lemma 3.3(b), for any $\delta' \in \Delta$, one has $\delta''' := \delta' + \delta'' - 1 \in \Delta$, it follows that

$$\delta' + \beta' - 1 = \begin{cases} \delta' + (\beta'' + \delta'') - 1 = \delta''' + \beta'' \in \bar{B} \\ \delta' + \delta'' - 1 \in \Delta \subseteq \bar{B} \\ \delta' + (\beta'' + 1) - 1 = \delta' + \beta'' \in \bar{B} \end{cases}$$

(iii) It follows from statement (c) of Lemma 3.3 that for any $\delta' \in \Delta$, one has $\delta' - 1 \in \Delta$ and as $\Delta \subseteq \bar{B}$, (c) then follows. □

Proof of Lemma 3.2. First we prove that for any

$$q(t, x) \sim \sum_{k=0}^{\infty} c_k^{\pm}(t)(\pm x)^{\delta_k} \ \text{ as } \ x \to \pm\infty \tag{41}$$

with exponents $\Delta = \{\delta_k\}_{k\geq 0}$, $\delta_0 = \delta > \delta_1 > ...$, satisfying claims (a)-(b) of Lemma 3.3, the initial value problem (35)-(36) with $p_0(x)$ satisfying (37) has a formal solution $\chi_+(t,x)$ given by ($t \in \mathbb{R}, x > 0$)

$$\chi_+(t,x) = \sum_{k=0}^{\infty} a_k^+(t) x^{\bar\beta_k+1} + a_*^+(t)\log x \tag{42}$$

where $\{\bar\beta_k + 1\}_{k\geq 0} = \bar B$ with $\bar\beta_0 > \bar\beta_1 > ....$ The existence of a formal solution $\chi_-(t,x)$ for $t \in \mathbb{R}$, $x < 0$ follows by the same arguments. Let us stress that the exponents $\beta_k + 1$ in the asymptotic expansion of the initial data $p_0(x)$ (cf. (37)) belong to the set $\{\beta' + 1 \mid \beta' \in B\}$ which is included in the larger set $\bar B$. Hence, the coefficients $a_k^+(0)$ in the asymptotic expansion (42) evaluated at $t = 0$ are zero or coincide with some of the constants p_k^+. Moreover $a_*^+(0) = p_*^+$.

Substituting (41) and (42) into (35) and using the notation $\dot{} = \frac{d}{dt}$ one obtains, in the case $x > 0$,

$$\begin{aligned}
\dot a_*^+(t)\log x + \dot a_0^+(t)x^{\beta+1} + \dot a_1^+(t)x^{\bar\beta_1+1} + \dot a_2^+(t)x^{\bar\beta_2+1} + ... = \\
= 2\Big(c_0^+(t)x^\delta + c_1^+(t)x^{\delta_1} + ...\Big)\Big(a_*^+(t)x^{-1} + \\
(\beta+1)a_0^+(t)x^\beta + (\bar\beta_1+1)a_1^+(t)x^{\bar\beta_1} + ...\Big) \\
-\Big(\delta c_0^+(t)x^{\delta-1} + \delta_1 c_1^+(t)x^{\delta_1-1} + ...\Big)
\end{aligned} \tag{43}$$

The maximal power of x on the right side of (43) is not bigger than $m_r = \max\{\beta+\delta, \delta-1\}$. As $\beta < 1/2$ and $\delta = \max\{2\beta, \beta-1\} < 1$ one obtains that $\beta + 1 > m_r$. Hence, $\dot a_0^+(t) = 0$ and thus $a_0^+(t) = a_0^+(0)$. Comparing the coefficients in (43) we also obtain that $\dot a_*^+(t) = 0$ and hence $a_*^+(t) = a_*^+(0)$.

Comparing the coefficients in (43) corresponding to terms of order $\bar\beta_k + 1$ in x one obtains that for any $k \geq 1$

$$\dot a_k^+(t) = P_k^+(a_0^+, a_1^+(t), ..., a_{k-1}^+(t)) + F_k^+(t) \tag{44}$$

where P_k^+ is a linear combination of the variables $a_0^+, ..., a_{k-1}^+$ with coefficients which are smooth functions of $t \in \mathbb{R}$. The term $F_k^+(t)$ is equal to $2c_{i_k}^+(t)p_*^+(0) - \delta_{i_k}c_{i_k}^+(t)$ iff there exists an index $i_k \geq 0$ such that $\bar\beta_k + 1 = \beta_{i_k} - 1$. If there is no such i_k then $F_k^+(t) \equiv 0$. Let us prove formula (44). It is clear from (43) that the right side of (44) is a sum of a linear polynomial of the variables $a_0^+, a_1^+, ...$ and an inhomogeneous term $F_k^+(t)$ of the form described above. Assume that there exists a_n^+, $n \geq k$, that enters as a linear term on the right side of (44). Then clearly there exists $m_n \geq 0$ such that

$$\bar\beta_n + \delta_{m_n} = \bar\beta_k + 1\ .$$

As $\bar\beta_n \leq \bar\beta_k$ and $\delta_{m_n} \leq \delta < 1$, it follows that $\bar\beta_n + \delta_{m_n} < \bar\beta_k + 1$. This contradiction proves (44).

Integrating equation (44) for $k = 1, 2, ...$ we find recursively the coefficients $a_k^+(t)$ in terms of the initial values $(a_i^+(0))_{0\leq i\leq k}$. Clearly, the formal solution

$\chi_+(t,x)$ satisfies (43) and by construction $\chi_+(0,x) = p_0(x)$. Arguing similarly we find a formal solution $\chi_-(t,x)$ for $t \in \mathbb{R}$, $x < 0$.

Next we show how the constructed formal solutions

$$\chi_\pm(t,x) = \sum_{k=0}^{\infty} a_k^\pm(t)(\pm x)^{\bar{\beta}_k+1} + a_*^\pm(t)\log(\pm x)$$

lead to a solution of (35)-(36). Choose $f(t,x) \in C^\infty(\mathbb{R}\times\mathbb{R})$ so that f has asymptotic expansions of the form

$$f(t,x) \sim \sum_{k=0}^{\infty} a_k^\pm(t)(\pm x)^{\bar{\beta}_k+1} + a_*^\pm(t)\log(\pm x) \quad \text{as} \quad x \to \pm\infty \tag{45}$$

with coefficients $(a_k^\pm(t))_{k\geq 0}$, $a_*^\pm(t)$ defined as above. The existence of such a function f follows, for example, from [12, Proposition 3.5]. Following [2, 1] we will call the function $f(t,x)$ an *asymptotic solution* of (35)-(36). Let $f_0 := f|_{t=0}$.

With the help of the asymptotic solution f we want to find a solution $p(t,x)$ of (35)-(36) of the form

$$p(t,x) := f(t,x) + s(t,x) \tag{46}$$

where $s(t,x) \in C^\infty(\mathbb{R}\times\mathbb{R})$ has to be determined so that s and all derivatives $\partial_t^l\partial_x^k s$ are fast decaying as $|x| \to \infty$. Substituting (46) into (35)-(36) one obtains the linear evolution equation

$$s_t(t,x) = 2q(t,x)s_x(t,x) + \eta(t,x) \tag{47}$$

$$s|_{t=0} = s_0(x) \tag{48}$$

where $\eta(t,x) := -f_t(t,x) + 2q(t,x)f_x(t,x) - q_x(t,x)$ belongs to $\mathcal{S}_{-\infty}(\mathbb{R}\times\mathbb{R})$ (as $f(t,x)$ is an asymptotic solution of (35)-(36)) and $s|_{t=0} = p_0(x) - f_0(x) \in \mathcal{S}(\mathbb{R})$, where as usual, $\mathcal{S}(\mathbb{R})$ denotes the functions of Schwartz class. By definition, $g \in C^\infty(\mathbb{R}\times\mathbb{R})$ belongs to the space $\mathcal{S}_{-\infty}(\mathbb{R}\times\mathbb{R})$ iff for any compact interval $J \subseteq \mathbb{R}$ and any $k,i,j \geq 0$ there exists a constant $C_{J,k,i,j} > 0$ such that for any $|x| \geq 1$ and $t \in J$

$$|\partial_t^i\partial_x^j g(t,x)| \leq C_{J,k,i,j}|x|^{-k}.$$

In particular, if $g \in \mathcal{S}_{-\infty}(\mathbb{R}\times\mathbb{R})$ then for any given $t \in \mathbb{R}$, the function $g(t,\cdot)$ belongs to $\mathcal{S}(\mathbb{R})$.

Due to Lemma A.3, we can find a solution $s \in \mathcal{S}_{-\infty}(\mathbb{R}\times\mathbb{R})$ of (47)-(48) which proves the existence part of Lemma 3.2. The uniqueness of the solution $p(t,x)$ in $C^\infty(\mathbb{R}\times\mathbb{R})$ follows from Lemma A.1. □

4 Proof of Theorem 1.2 and Theorem 1.3

In this section we prove the global (in time) existence and the uniqueness of solutions of the mKdV equation stated in Theorem 1.2 and Theorem 1.3.

Before proving these theorems we introduce the following auxiliary spaces

$$\mathcal{O}^*_{\beta+1}(\mathbb{R}\times\mathbb{R}) := \{f \in C^\infty(\mathbb{R}\times\mathbb{R}) \mid f_x \in \mathcal{O}_\beta(\mathbb{R}\times\mathbb{R})\}$$

and

$$o^*_{\beta+1}(\mathbb{R}\times\mathbb{R}) := \{f \in C^\infty(\mathbb{R}\times\mathbb{R}) \mid f_x \in o_\beta(\mathbb{R}\times\mathbb{R})\}\,.$$

Proof of Theorem 1.2. We follow the arguments in the proof of Theorem 1.1. Given $r_0 \in \mathcal{O}_\beta(\mathbb{R})$ we get $q_0 := r_0' + r_0^2$ which belongs to the space $\mathcal{O}_\delta(\mathbb{R})$ with $\delta := \max\{2\beta, \beta-1\} < 1$. By Theorem 2 in [4] (cf. Theorem B.3, Appendix B) there exists a solution $q \in \mathcal{O}_\delta(\mathbb{R}\times\mathbb{R})$ of the KdV initial value problem

$$q_t - 6qq_t + q_{xxx} = 0, \quad q|_{t=0} = q_0\,.$$

As $\delta < 1$ the solution $q(t,x)$ satisfies the growth condition (14). Let $\psi(t,x) > 0$ be the globally defined unique solution of (33)-(34) (see also Proposition 2.2). According to Theorem 2.5 (a) the function $r(t,x) = \psi_x(t,x)/\psi(t,x)$ is a solution of the mKdV initial value problem (1)-(2). Then $p(t,x) := \log\psi(t,x)$ satisfies (35)-(36) with $p_0(x) = \int_0^x r_0(s)\, ds$. As p_0 is in $\mathcal{O}^*_{\beta+1}(\mathbb{R})$ the solution $p(t,x)$ of (35)-(36) belongs to $\mathcal{O}^*_{\beta+1}(\mathbb{R}\times\mathbb{R})$ (cf. Lemma 4.1 below). In particular $r(t,x) = p_x(t,x) \in \mathcal{O}_\beta(\mathbb{R}\times\mathbb{R})$.

The uniqueness of the solution $r(t,x)$ constructed above follows from Theorem 2.5 (b) and the uniqueness result for KdV in Theorem 1 in [3] (cf. Theorem B.2, Appendix B). $\square$

In the proof of Theorem 1.2 we used the following analogue of Lemma 3.2.

Lemma 4.1. *Let $\beta < 1/2$ and $\delta := \max\{2\beta, \beta-1\} < 1$. Consider the initial value problem* (35)-(36) *where $q \in \mathcal{O}_\delta(\mathbb{R}\times\mathbb{R})$. Then for any initial data $p_0 \in \mathcal{O}^*_{\beta+1}(\mathbb{R})$ there exists a solution $p \in \mathcal{O}^*_{\beta+1}(\mathbb{R}\times\mathbb{R})$ of* (35)-(36) *which is unique in $C^\infty(\mathbb{R}\times\mathbb{R})$.*

Proof of Lemma 4.1. The lemma is proved by the same arguments as the ones used in the proof of Lemma A.3 (see also [4], Proposition 1). $\square$

Proof of Theorem 1.3. The proof is similar to the proof of Theorem 1.2 and is based on the existence and uniqueness results for the initial value problem of KdV of [3, 4] (cf. Theorem B.2, B.4 in Appendix B) and on a variant of Lemma 4.1 where the spaces $\mathcal{O}_\beta$ and $\mathcal{O}^*_{\beta+1}$ are replaced by the spaces o_β and $o^*_{\beta+1}$ respectively. $\square$

We conclude this section by stating a more general uniqueness result for the mKdV initial value problem (1)-(2). Let $I = (a,b) \subseteq \mathbb{R}$ with $-\infty \le a < b \le \infty$. Denote by $\mathcal{G}(\mathbb{R}\times\mathbb{R})$ the linear space of functions $r(t,x)$ in $C^\infty(\mathbb{R}\times\mathbb{R})$ such that for any compact interval $J \subseteq I$ one has for $|x| \ge 1$ and any $k \ge 1$

$$r(t,x) = o(\sqrt{|x|}) \quad \text{and} \quad \partial_x^k r(t,x) = O(1/\sqrt{|x|}),$$

uniformly in $t \in J$. The following theorem follows in a straightforward way from Theorem 2.5 (b) and Theorem 1 in [3] (cf. Theorem B.2, Appendix B).

Theorem 4.2. *There exists at most one solution of the mKdV initial value problem (1)-(2) in $\mathcal{G}(\mathbb{R}\times\mathbb{R})$.*

A Appendix A

In this appendix we state and prove, for the convenience of the reader, a result on the first order linear PDE, used in the main body of the paper,

$$\begin{aligned} u_t(t,x) &= a(t,x)u_x(t,x) + b(t,x)u(t,x) && (49)\\ u|_{t=0} &= \psi(x) && (50)\end{aligned}$$

where $\psi \in C^\infty(\mathbb{R})$, $a, b \in C^\infty(\mathbb{R}\times\mathbb{R})$, and a grows for $x \to \pm\infty$ at most linearly. In addition, we prove two technical lemmas used in the proof of Theorem 1.1.

Lemma A.1. *Assume that for any $T > 0$ there exists a constant $C_T > 0$ such that for any $|x| \geq 1$*

$$|a(t,x)| \leq C_T|x| \tag{51}$$

uniformly for $t \in [-T,T]$. Then

(a) for any initial datum $\psi \in C^\infty(\mathbb{R})$ there exists a unique global in time solution $u \in C^\infty(\mathbb{R}\times\mathbb{R})$;

(b) if $\psi(x) > 0$ $\forall x \in \mathbb{R}$ then $u(t,x) > 0$ $\forall t, x \in \mathbb{R}$.

Proof. Clearly, the equation (49) can be rewritten in the form

$$X(u) = bu \tag{52}$$

where $X := \partial_t - a\partial_x$. Consider the ordinary differential equation

$$\dot{x} = -a(t,x), \tag{53}$$

$$x|_{t=0} = x_0. \tag{54}$$

It follows from (51) that if a solution $x(t,x_0)$ of (53)-(54) is defined on the interval $t \in (-T,T)$ for some $0 < T < \infty$ then it satisfies the a priory estimate

$$\sup_{|t|<T} |x(t,x_0)| < (1+|x_0|)\, e^{C_T T}.$$

In particular, the latter estimate implies that for any $x_0 \in \mathbb{R}$, there exists a unique global (in time) solution $x(t,x_0)$ of (53)-(54). To prove uniqueness of a solution of (49)-(50), assume that $u = u(t,x)$ is a smooth solution. It follows from (52) that for any $x_0 \in \mathbb{R}$, the function $v(t) := u(t,x(t))$ with $x(t) := x(t,x_0)$ satisfies the differential equation $\dot{v}(t) = b(t,x(t))v(t)$, hence

$$u(t,x(t)) = \psi(x_0)e^{\int_0^t b(s,x(s))\,ds}. \tag{55}$$

As for any given $t \in \mathbb{R}$, the transformation $\mathbb{R} \to \mathbb{R}, x_0 \mapsto x(t,x_0)$, is a diffeomorphism, formula (55) defines $u(t,x)$ uniquely. At the same time, (55) defines a smooth global in time solution of (49)-(50). This proves claim (a). Claim (b) also follows from (55). $\square$

Remark A.2. *Let $a(t,x) = |x|^\alpha$ for $|x| \geq 1$, where $\alpha > 1$. Then solutions of (53) can blow up in finite time. Moreover, one can show that a solution of* (49)-(50) *is not necessarily unique. This shows that assumption (51) is essential claim* (a) *to be true.*

In the remainder of this appendix, we prove, as advertised, two technical lemmas used in the proof of Theorem 1.1. As above, $\mathcal{S}(\mathbb{R})$ denotes the space of functions $f : \mathbb{R} \to \mathbb{R}$ of Schwartz class.

Lemma A.3. *Assume that $q(t,x) \in \mathcal{S}_\delta(\mathbb{R} \times \mathbb{R})$ with $\delta < 1$. Then the initial value problem (47)-(48) with $\eta \in \mathcal{S}_{-\infty}(\mathbb{R} \times \mathbb{R})$ and $s_0 \in \mathcal{S}(\mathbb{R})$ has a solution in $\mathcal{S}_{-\infty}(\mathbb{R} \times \mathbb{R})$ which is unique in $C^\infty(\mathbb{R} \times \mathbb{R})$.*

Proof. The initial value problem (47)-(48) can be rewritten as

$$X(s) = \eta \tag{56}$$

$$s|_{t=0} = s_0 \tag{57}$$

where $X(t,x) := \partial_t - 2q(t,x)\partial_x$ and $X(s)$ denotes the derivative of s with respect to the flow of the vector field X. Denote by $\xi(t; t_0, x_0)$ the solution of the ordinary differential equation

$$\dot{x} = -2q(t,x), \tag{58}$$

$$x|_{t=t_0} = x_0. \tag{59}$$

If $t_0 = 0$ we denote the corresponding solution $\xi(t; 0, x_0)$ by $\xi(t, x_0)$. As $q \in \mathcal{S}_\delta(\mathbb{R} \times \mathbb{R})$ and $\delta < 1$ it follows that for any $0 < T < \infty$ there exists $C_T > 0$ such that for any $|x| \geq 1$ and $t \in [-T, T]$

$$|q(t,x)| \leq C_T |x| \, . \tag{60}$$

In particular, (60) implies that the solution $\xi(t; t_0, x_0)$ is defined for any $t \in \mathbb{R}$. As $q(t,x)$ is C^∞-smooth in (t,x), the solution $\xi(t; t_0, x_0)$ is unique and depends smoothly on the initial data (t_0, x_0). Moreover, for any given $t_0, t \in \mathbb{R}$, $t \geq t_0$, the transformation $x_0 \mapsto \xi(t; t_0, x_0)$, $\mathbb{R} \to \mathbb{R}$, is a diffeomorphism. Let $s(t,x)$ be a smooth solution of (56)-(57). Then the function $s(t) := s(t, \xi(t, x_0))$ satisfies the differential equation $\dot{s} = \eta(t, \xi(t, x_0))$. In particular,

$$s(t, \xi(t, x_0)) = s_0(x_0) + \int_0^t \eta(\tau, \xi(\tau, x_0)) \, d\tau. \tag{61}$$

Hence, the smooth solution $s(t,x)$ of (56)-(57) is defined uniquely by the right side of (61). Equation (61) can be rewritten in the form

$$s(t,x) = s_0(\xi(0; t, x)) + \int_0^t \eta(\tau, \xi(\tau; t, x)) \, d\tau \, . \tag{62}$$

Using that $s_0 \in \mathcal{S}(\mathbb{R})$, $\eta \in \mathcal{S}_{-\infty}(\mathbb{R} \times \mathbb{R})$ together with (62) and Lemma A.4 (a) stated below one easily gets that for any $0 < T < \infty$ and for any $k \geq 0$ there exists a constant $C_{T,k} > 0$ such that for any $t \in [-T, T]$ and any x with $|x| \geq 1$

$$|s(t,x)| \leq C_{T,k}|x|^{-k} \ .$$

Differentiating equation (62) with respect to t and x we obtain that for any $k, l \geq 0$, the partial derivative $\partial_t^k \partial_k^l s(t,x)$ is a finite sum

$$\partial_t^k \partial_k^l s(t,x) = \sum_j S_j(t,x)$$

where the terms $S_j(t,x)$, with the help of Lemma A.4 below, can be shown to be of the form $S_j(t,x) = P_j(t,x)Q_j(t,x)$ with $P_j \in \mathcal{S}_{-\infty}(\mathbb{R}\times\mathbb{R})$ and Q_j growing at most polynomially in x uniformly on compact sets of t. In particular, we get that the solution $s(t,x)$ of the initial value problem (47)-(48) lies in $\mathcal{S}_{-\infty}(\mathbb{R}\times\mathbb{R})$. The uniqueness of the solution follows from the same arguments as in in the proof of Lemma A.1. $\square$

The following lemma is used in the proof of Lemma A.3. We use the same notation as in the proof of this lemma.

Lemma A.4. *Assume that $q(t,x) \in \mathcal{S}_\delta(\mathbb{R} \times \mathbb{R})$ with $\delta < 1$. Then the following statements hold:*

(a) For any $0 < T < \infty$ there exist constants $C_1 = C_1(T)$, $C_2 = C_2(T)$, $0 < C_1 < C_2$, and $N = N(T) > 0$ such that for any $t, t' \in [-T, T]$ and x with $|x| \geq N$

$$C_1|x| \leq |\xi(t;t',x)| \leq C_2|x| \ . \tag{63}$$

(b) For any $0 < T < \infty$ and for any $k, l, m \geq 0$ with $k + l \geq 1$, there exists a constant $C_{T,k,l,m} > 0$ such that for any $t, t' \in [-T, T]$ and x with $|x| \geq 1$

$$|\partial_t^k \partial_{t'}^l \partial_x^m \xi(t;t',x)| \leq C_{T,k,l,m}|x|^{\delta - m} \ . \tag{64}$$

(c) For any $0 < T < \infty$ and for any $m \geq 0$ there exists a constant $C_{T,m} > 0$ such that for any $t, t' \in [-T, T]$ and x with $|x| \geq 1$

$$|\partial_x^m \xi(t;t',x)| \leq C_{T,m}|x|^{1-m} \ . \tag{65}$$

Proof. Let $R(t,x) := -2q(t,x)$. Clearly,

$$R \in \mathcal{S}_\delta(\mathbb{R} \times \mathbb{R}), \quad \delta < 1. \tag{66}$$

(a) As $R(t,x)$ satisfies for any given $0 < T < \infty$ the growth condition (60) for $x \geq 1$ and $|t| \leq T$ with some constant $C_T > 0$, the solution $\xi(t;t',x)$ (defined globally in time) of the ordinary differential equation

$$\dot{\xi} = R(t,\xi) \tag{67}$$

$$\xi|_{t=t'} = x \tag{68}$$

satisfies for any $x \geq 1$ and $t, t' \in [-T, T]$,

$$-C_T \leq \dot{\xi}/\xi \leq C_T$$

or

$$xe^{-C_T|t-t'|} \leq \xi(t; t', x) \leq xe^{C_T|t-t'|}.$$

Hence, for any $x \geq N := e^{2C_T T}$ and $t, t' \in [-T, T]$ one has $xe^{-2C_T T} \leq \xi(t; t', x) \leq xe^{2C_T T}$. Similarly one argues for $x \leq -N$ to conclude, altogether, that

$$e^{-2C_T T}|x| \leq |\xi(t; t', x)| \leq e^{2C_T T}|x|$$

for any $t, t' \in [-T, T]$ and any $|x| \geq N$.

(b) First define a class of continuous functions $\mathcal{B}^\delta \equiv \mathcal{B}^\delta(\mathbb{R}^3)$. By definition, a continuous function $f : \mathbb{R}^3 \to \mathbb{R}$ is an element in $\mathcal{B}^\delta(\mathbb{R}^3)$ iff for any $0 < T < \infty$ there exists a constant $C_T > 0$ such that for any $t, t' \in [-T, T]$ and $|x| \geq 1$

$$|f(t, t', x)| \leq C_T |x|^\delta.$$

We start by proving that for any $k, l, m \geq 0$ with $k + l \geq 1$ the function $\partial_t^k \partial_{t'}^l \partial_x^m \xi(t; t', x)$ belongs to $\mathcal{B}^\delta$. For this purpose it is convenient to consider instead of (67)-(68) the ordinary differential equation

$$\dot{y} = R(t + t', y) \tag{69}$$

$$y|_{t=0} = x \tag{70}$$

where we consider $t' \in \mathbb{R}$ as a parameter. Clearly,

$$y(t; t', x) = \xi(t + t'; t', x) \tag{71}$$

where $\xi(t; t', x)$ is the solution of (67)-(68). Hence $\partial_t^k \partial_{t'}^l \partial_x^m \xi(t; t', x) \in \mathcal{B}^\delta$ if and only if

$$\partial_t^k \partial_{t'}^l \partial_x^m y(t; t', x) \in \mathcal{B}^\delta. \tag{72}$$

We will prove (72). As by assumption $R \in \mathcal{S}_\delta(\mathbb{R} \times \mathbb{R})$, the equation (69) together with the lower and upper bounds in (63) imply that $y_t(t; t', x) \in \mathcal{B}^\delta$.

Differentiating equation (69)-(70) with respect to x we obtain that $y_x(t; t', x)$ satisfies the differential equation

$$(y_x)_t = R_x(t + t', y) y_x \ , \tag{73}$$

$$y_x|_{t=0} = 1 \tag{74}$$

hence,

$$y_x(t; t', x) = e^{\int_0^t R_x(\tau + t', y(\tau; t', x))\, d\tau} \ . \tag{75}$$

As $R_x \in \mathcal{S}_{\delta-1}(\mathbb{R} \times \mathbb{R})$ with $\delta - 1 < 0$ we get from claim (a) that for any $0 < T < \infty$ there exists a constant $C_T > 0$ such that $\forall t, t' \in [-T, T]$ and any $x \in \mathbb{R}$ one has that

$$|y_x(t; t', x)| \leq C_T \ . \tag{76}$$

Analogously, differentiating (69)-(70) with respect to the variable t' one gets

$$\begin{aligned} (y_{t'})_t &= R_x(t+t',y)y_{t'} + R_t(t+t',y)\ , & (77)\\ y_{t'}|_{t=0} &= 0\ . & (78) \end{aligned}$$

By the method of the variation of parameters one obtains that

$$y_{t'}(t;t',x) = \Big(\int_0^t b(\tau)e^{-\int_0^\tau a(u)\ du}\ d\tau\Big)e^{\int_0^t a(u)\ du} \qquad (79)$$

where $a(t) = a(t,t',x) := R_x(t+t',y)$ and $b(t) = b(t,t',x) := R_t(t+t',y)$. As $R_x \in \mathcal{S}_{\delta-1}(\mathbb{R}\times\mathbb{R})$ and $R_t \in \mathcal{S}_\delta(\mathbb{R}\times\mathbb{R})$ we get that $a \in \mathcal{B}^{\delta-1}$, $b \in \mathcal{B}^\delta$. Using (79) and $\delta - 1 < 0$ one concludes that $y_{t'} \in \mathcal{B}^\delta$. Differentiating successively (69)-(70) with respect to the variables t' and x one obtains an equation of the form

$$(\partial_{t'}^l\partial_x^m y)_t = R_x(t+t',y)(\partial_{t'}^l\partial_x^m y) + B(t,t',x)$$

where the inhomogeneous term B is an element in $\mathcal{B}^{\delta-m}$. Hence arguing as above one concludes that $\partial_{t'}^l\partial_x^m y(t;t',x) \in \mathcal{B}^{\delta-m}$.

In order to prove that

$$\partial_t^k(\partial_{t'}^l\partial_x^m y(t;t',x)) \in \mathcal{B}^{\delta-m} \qquad (80)$$

for any $k \geq 0$ we use induction in k. By the considerations from above (80) holds for $k = 0$. Assume that $k \geq 1$ and $\partial_t^j\partial_{t'}^l\partial_x^m y \in \mathcal{B}^{\delta-m}$ for $0 \leq j \leq k-1$. Differentiating equation (69) with respect to t', x, and t we obtain

$$\begin{aligned} (\partial_t^{k-1}\partial_{t'}^l\partial_x^m y)_t &= \partial_t^{k-1}\partial_{t'}^l\partial_x^m(R(t+t',y))\\ &= R_x(t+t',y)(\partial_t^{k-1}\partial_{t'}^l\partial_x^m y) + B(t,t',x). & (81) \end{aligned}$$

Using (63) once again together with the induction hypothesis one proves that the inhomogeneous term B is in $\mathcal{B}^{\delta-m}$. As $R_x \in \mathcal{B}^{\delta-1}$ with $\delta - 1 < 0$ and $\partial_t^{k-1}\partial_{t'}^l\partial_x^m y \in \mathcal{B}^{\delta-m}$ by induction hypothesis, formula (81) implies (80).

(c) Statement (c) follows from (63) (for $m = 0$), (76) (for $m = 1$) and then by differentiating (75) and using that $R_{xx} \in \mathcal{S}_{\delta-2}(\mathbb{R}\times\mathbb{R})$ (for $m \geq 2$). □

Remark A.5. *A result similar to Lemma A.4 with a non-linear term in the equation, can be found in [4].*

B Appendix B

For the convenience of the reader, we state in this appendix the results on the existence and uniqueness of solutions of the KdV equation

$$q_t - 6qq_x + q_{xxx} = 0 \qquad (82)$$

$$q|_{t=0} = q_0 \qquad (83)$$

proved by Bondareva and Shubin [1, 2, 3, 4] which we use in the main body of the paper.

Following earlier work of Menikoff [10], the authors of [2, 1] prove (among other things) the following theorem:

Theorem B.1. *For any $\beta < 1$ and for any initial data $q_0 \in \mathcal{S}_\beta(\mathbb{R})$ there exists a global (in time) solution $q \in \mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$ of the initial value problem (82)-(83).*

Completing results of Menikoff [10] the following uniqueness theorem is proved in [3], by use of a version of Holmgren's principle.

Theorem B.2. *For any $T > 0$, there is at most one solution of (82)-(83) in the classes of functions $q \in C^\infty([0,T] \times \mathbb{R})$ such that*

$$q(t,x) = o(|x|) \quad \text{and} \quad \partial_x^k q(t,x) = O(1) \quad \forall k \geq 1$$

uniformly in $t \in [0,T]$.

In [4] the following theorems are proved:

Theorem B.3. *For any $\beta < 1$ and for any initial data $q_0 \in \mathcal{O}_\beta(\mathbb{R})$ there exists a global in time solution $q \in \mathcal{O}_\beta(\mathbb{R} \times \mathbb{R})$ of the initial value problem (82)-(83).*

Theorem B.4. *For any $\beta \leq 1$ and for any initial data $q_0 \in o_\beta(\mathbb{R})$ there exists a global in time solution $q \in o_\beta(\mathbb{R} \times \mathbb{R})$ of the initial value problem (82)-(83).*

Remark B.5. *According to Theorem B.2 the solutions in Theorem B.1, B.3, and B.4 are unique in the corresponding classes.*

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